Transport coefficients, effective charge and mass for multicomponent systems with fractional exclusion statistics

1995 J. Phys. A: Math. Gen. 28 6027

(http://iopscience.iop.org/0305-4470/28/21/010)

J. Phys. A: Math. Gen. 28 (1995) 6027–6038. Printed in the UK

Transport coefficients, effective charge and mass for multicomponent systems with fractional exclusion statistics

Takahiro Fukui†‡ and Norio Kawakami
Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto 606-01, Japan

Received 24 April 1995

Abstract. Transport properties of the multicomponent quantum many-body systems obeying Haldane's fractional exclusion statistics are studied in one dimension. By computing the finite-size spectrum under twisted boundary conditions, we explicitly express the conductivity and conductance in terms of statistical interactions. Through this analysis, the effective charge and effective mass for collective excitations are determined. We apply the results for $1/r^2$ quantum systems as well as for correlated electron systems.

1. Introduction

In low-dimensional quantum systems, excitations are described by quasiparticles carrying fractional quantum numbers. One of the well known examples is the fractional quantum Hall effect (FQHE) [1], where quasiparticles are classified by the fractional charge and statistics. In these theories, fractional quantum numbers arise from exchange properties of the wavefunction. Recently, Haldane [2] proposed a new concept of fractional statistics based on the state-counting of many-body systems, which is a generalization of the Pauli exclusion principle. We will refer to this as fractional exclusion statistics. The thermodynamic properties have already been investigated in detail [3–6]. For example, Wu and Bernard [4] formulated thermodynamic equations, and showed that the statistical interaction is related to the two-body phase shift for Bethe-ansatz solvable models. Their method was generalized to multicomponent systems [6], and low-energy critical properties were investigated.

In this paper, we study transport properties of multicomponent quantum systems with exclusion statistics in one dimension. Transport coefficients are closely related to wavefunctions, and may usually be calculated in the Green function formalism. Since we have only statistics between particles without explicit wavefunctions, it is not trivial to study such properties directly from the definition of statistics [7]. We therefore use a trick to avoid this difficulty. Namely, by applying the idea of twisted boundary conditions for the finite-size spectrum, we calculate the conductivity and the conductance in terms of statistical interactions. We then determine the effective charge and the effective mass, and show how these quantities are related to the exclusion statistics.

After a brief introduction of exclusion statistics in section 2, we compute the finite-size corrections due to the vector potential for multicomponent quantum systems in section 3. In

† JSPS Research Fellow.
‡ E-mail address: fukui@yukawa.kyoto-u.ac.jp

0305-4470/95/216027+12$19.50

section 4 we then obtain the conductivity and the conductance, and determine the effective charge and the effective mass. In section 5 we apply the results for several interesting quantum systems. Section 6 is devoted to a summary and discussions.

2. Exclusion statistics

Let us start with a brief introduction for fundamental properties of exclusion statistics [2, 4]. It is based on counting the change of the dimension of the one-particle Hilbert space when a particle is added to the system, which is explicitly formulated as

$$\frac{\partial D_\alpha(k_\alpha)}{\partial N_\beta(k'_\beta)} = -g_{\alpha\beta}(k_\alpha - k'_\beta) \equiv -\{g'_{\alpha\beta}(k_\alpha - k'_\beta) + \delta_{\alpha\beta}\delta_{k_\alpha k'_\beta}\} \tag{2.1}$$

where $D_\alpha(k_\alpha)$ and $N_\alpha(k_\alpha)$ are, roughly speaking, the numbers of unoccupied (hole) and occupied (particle) states specified by the internal quantum numbers $\alpha = (1, 2, \ldots, M)$ and corresponding momenta k_α. The matrix $g_{\alpha\beta}$, which is called the *statistical interaction*, describes the correlation effects between particles. See [2] for a more rigorous definition. The simple cases $g_{\alpha\beta}(k_\alpha - k'_\beta) = g\delta_{\alpha\beta}\delta_{k_\alpha k'_\beta}$ with $g = 1$ and $g = 0$ correspond to free fermions and free bosons, respectively, and for the general fractional value g, we call it *ideal fractional exclusion statistics*.

The statistical interactions should be independent of N_α [2], and hence equation (2.1) results in

$$D_\alpha(k_\alpha) = -\sum_{\beta,k'_\beta} g_{\alpha\beta}(k_\alpha - k'_\beta)N_\beta(k'_\beta) + D^0_\alpha(k_\alpha). \tag{2.2}$$

We assume that the integral constants are given by $D^0_\alpha(k_\alpha) = D^0\delta_{\alpha 1}$ or D^0, which are referred to as hierarchical and symmetric bases, respectively. Such bases were originally used for a classification of the FQHE [8, 9]. Also, in one-dimensional quantum systems the hierarchical basis serves as a natural basis for the Bethe-ansatz solution [10].

In the thermodynamic limit, we introduce the distribution functions for particles and holes:

$$\rho_\alpha(k_\alpha) = \frac{N_\alpha(k_\alpha)}{D^0} \qquad \rho^{(h)}_\alpha(k_\alpha) = \frac{D_\alpha(k_\alpha)}{D^0} \tag{2.3}$$

where D^0 is proportional to the system size L such that $D^0 = L/2\pi$ under periodic boundary conditions [4]. The bare charge for each elementary excitation is defined by [6]

$$t_\alpha = \frac{D^0_\alpha}{D^0}. \tag{2.4}$$

Consequently, equation (2.2) can be written as

$$\rho_\alpha(k_\alpha) + \rho^{(h)}_\alpha(k_\alpha) = t_\alpha - \sum_\beta \int_{-\infty}^{\infty} \mathrm{d}k'_\beta \; g'_{\alpha\beta}(k_\alpha - k'_\beta)\rho_\beta(k'_\beta) \tag{2.5}$$

The energy of the system is assumed to take the form [4]:

$$\varepsilon \equiv \frac{E}{D^0} = \sum_\alpha \int_{-\infty}^{\infty} \mathrm{d}k_\alpha \; \epsilon^0_\alpha(k_\alpha)\rho_\alpha(k_\alpha) \tag{2.6}$$

with the bare energy function $\epsilon^0_\alpha(k)$. Note that many-body effects due to the exclusion statistics are incorporated in the distribution function $\rho(k)$.

The thermodynamic equations are generally obtained in a set of coupled nonlinear equations at finite temperatures†. Here we restrict ourselves to the zero-temperature case which is sufficient for the following calculation. At zero temperature without external fields, M species of elementary excitations are specified by the dressed energy function $\epsilon_\alpha(k)$. The 'Fermi level' Q_α for each excitation is determined by the conditions, $\epsilon_\alpha(k_\alpha) < 0$ for $|k_\alpha| < Q_\alpha$ and $\epsilon_\alpha(k_\alpha) > 0$ for $|k_\alpha| > Q_\alpha$, provided that the energy dispersion may be symmetric around the origin $k_\alpha = 0$. Equation (2.5) then reduces to the following integral equations supplemented by those for the dressed energy ϵ_α:

$$\rho_\alpha(k_\alpha) = t_\alpha - \sum_\beta \int_{-Q_\beta}^{Q_\beta} \mathrm{d}k'_\beta \; g'_{\alpha\beta}(k_\alpha - k'_\beta)\rho_\beta(k'_\beta) \tag{2.7}$$

$$\epsilon_\alpha(k_\alpha) = \epsilon^0_\alpha(k_\alpha) - \mu t_\alpha - \sum_\beta \int_{-Q_\beta}^{Q_\beta} \mathrm{d}k'_\beta \; g'_{\beta\alpha}(k'_\beta - k_\alpha)\epsilon_\beta(k'_\beta). \tag{2.8}$$

The total energy is now expressed by the dressed energy as

$$\varepsilon = \sum_\alpha \varepsilon_\alpha + \mu n_c \qquad \varepsilon_\alpha = t_\alpha \int_{-Q_\alpha}^{Q_\alpha} \mathrm{d}k_\alpha \; \epsilon_\alpha(k_\alpha) \tag{2.9}$$

where μ is the chemical potential and n_c is the density of charged particles. These equations can describe the static properties at zero temperature.

3. Finite-size corrections due to static vector potential

We now turn to the computation of the conductivity in pure systems without randomness, which can be calculated with the response to the vector potential. Consider the ring system at $T = 0$ threaded by the magnetic flux, which gives rise to a static vector potential along the ring [11]. The effect of the vector potential is incorporated in twisted boundary conditions [12, 13]. Therefore, the energy increment quadratically proportional to A can be calculated through the analysis of the finite-size spectrum, which directly gives the charge stiffness and hence the conductivity. By observing that the basic equations (2.7) and (2.8) have the same structure as the Bethe-ansatz equations, we can apply the elegant techniques of the dressed charge matrix developed for integrable models [14], and generalize the calculation of the conductivity [12, 13] to multicomponent cases.

Before proceeding with the finite-size corrections, we give here a formal solution to (2.7) in the absence of external fields, which is necessary for the following discussions. To this end, let us first introduce the functions [14, 15]

$$K_{\alpha\beta}(k_\alpha - k'_\beta) = g'_{\alpha\beta}(k_\alpha - k'_\beta) - \sum_\gamma \int_{-Q_\gamma}^{Q_\gamma} \mathrm{d}k''_\gamma \; g'_{\alpha\gamma}(k_\alpha - k''_\gamma)K_{\gamma\beta}(k''_\gamma - k'_\beta) \tag{3.1}$$

$$Z_{\alpha\beta}(k_\beta) = \delta_{\alpha\beta} - \sum_\gamma \int_{-Q_\gamma}^{Q_\gamma} \mathrm{d}k'_\gamma \; Z_{\alpha\gamma}(k'_\gamma)g'_{\gamma\beta}(k'_\gamma - k_\beta) \tag{3.2}$$

† The thermodynamics of systems with exclusion statistics can be formulated by the method proposed in [4], and a multicomponent generalization can be found in [6]. A key idea is to introduce the entropy $S = \ln W$ with

$$W = \prod_{\alpha,k_\alpha} \frac{(D_\alpha + N_\alpha - 1)!}{N_\alpha!(D_\alpha - 1)!}$$

which plausibly interpolates the boson and fermion cases [2, 4]. Note that if we choose $g_{\alpha\beta}$ for free fermions and bosons, ρ in (2.3) reduces to the Fermi and Bose distribution functions.

where $Z_{\alpha\beta}$ is called the dressed charge matrix [14]. By noting the relation

$$Z_{\alpha\beta}(k_\beta) = \delta_{\alpha\beta} - \int_{-Q_\alpha}^{Q_\alpha} \mathrm{d}k'_\alpha \; K_{\alpha\beta}(k_\alpha - k'_\beta) \tag{3.3}$$

we can formally write down the distribution function in terms of the dressed charge matrix as

$$\rho_\alpha(k_\alpha) = \sum_\beta t_\beta Z_{\beta\alpha}(k_\alpha). \tag{3.4}$$

We now evaluate finite-size corrections to the total energy due to the vector potential, which are the same as those for twisted boundary conditions [12, 13]. First, by observing that the effect of the static vector potential A is to shift the momentum by the amount δ_α proportional to A (in units of e), the basic equations should read

$$\tilde{\rho}_\alpha(k_\alpha) = t_\alpha - \sum_\beta \int_{-Q_\beta+\delta_\beta}^{Q_\beta+\delta_\beta} \mathrm{d}k'_\beta \; g'_{\alpha\beta}(k_\alpha - k'_\beta)\tilde{\rho}_\beta(k'_\beta) \tag{3.5}$$

$$\tilde{\epsilon}_\alpha(k_\alpha) = \epsilon^0_\alpha(k_\alpha) - \mu t_\alpha - \sum_\beta \int_{-Q_\beta+\delta_\beta}^{Q_\beta+\delta_\beta} \mathrm{d}k'_\beta \; g'_{\beta\alpha}(k'_\beta - k_\alpha)\tilde{\epsilon}_\beta(k'_\beta) \tag{3.6}$$

$$\tilde{\varepsilon} = \sum_\alpha \int_{-Q_\alpha+\delta_\alpha}^{Q_\alpha+\delta_\alpha} \mathrm{d}k_\alpha \; \epsilon^0(k_\alpha)\tilde{\rho}_\alpha(k_\alpha) = \sum_\alpha \tilde{\varepsilon}_\alpha + \mu n_\mathrm{c} \tag{3.7}$$

where

$$\tilde{\varepsilon}_\alpha = t_\alpha \int_{-Q_\alpha+\delta_\alpha}^{Q_\alpha+\delta_\alpha} \mathrm{d}k_\alpha \; \tilde{\epsilon}(k_\alpha). \tag{3.8}$$

It should be noted that the vector potential not only shifts the momentum uniformly, but can also rearrange in general the distribution of the momentum via interactions between particles. For clarity, we indicate rearranged quantities by a tilde.

Let us compute the corrections to the total energy (3.7). By differentiating equation (3.6) with respect to δ, we have

$$\left.\frac{\partial\tilde{\epsilon}_\alpha(k_\alpha)}{\partial\delta_\beta}\right|_{\delta=0} = 0 \tag{3.9}$$

by using the conditional equation for the 'Fermi level', $\epsilon_\alpha(Q_\alpha) = 0$. By differentiating again, we have

$$\left.\frac{\partial^2\tilde{\epsilon}_\alpha(k_\alpha)}{\partial\delta_\beta\partial\delta_\gamma}\right|_{\delta=0} = -\epsilon'_\beta(Q_\beta)\{g'_{\beta\alpha}(Q_\beta - k_\alpha) + g'_{\beta\alpha}(-Q_\beta - k_\alpha)\}\delta_{\beta\gamma}$$

$$-\sum_\gamma \int_{-Q_\gamma}^{Q_\gamma} \mathrm{d}k'_\gamma \; g'_{\alpha\gamma}(k_\alpha - k'_\gamma) \left.\frac{\partial^2\tilde{\epsilon}_\gamma(k'_\gamma)}{\partial\delta_\beta\partial\delta_\gamma}\right|_{\delta=0}$$

which can be solved formally by the iteration scheme, resulting in

$$\left.\frac{\partial^2\tilde{\epsilon}_\alpha(k_\alpha)}{\partial\delta_\beta\partial\delta_\gamma}\right|_{\delta=0} = -\epsilon'_\beta(Q_\beta)\{K_{\beta\alpha}(Q_\beta - k_\alpha) + K_{\beta\alpha}(-Q_\beta - k_\alpha)\}\delta_{\beta\gamma} \tag{3.10}$$

where the prime in ϵ'_α stands for the derivative with respect to k_α. In what follows, we assume the relation $g_{\alpha\beta}(k_\alpha - k'_\beta) = g_{\beta\alpha}(k'_\beta - k_\alpha)$. Using these results, we have

$$\left.\frac{\partial\tilde{\varepsilon}_\alpha}{\partial\delta_\beta}\right|_{\delta=0} = 0 \tag{3.11}$$

$$\left.\frac{\partial^2\tilde{\varepsilon}_\alpha}{\partial\delta_\beta\partial\delta_\gamma}\right|_{\delta=0} = t_\alpha\epsilon'_\beta(Q_\beta)\left[2\delta_{\alpha\beta} - \int_{-Q_\alpha}^{Q_\alpha} \mathrm{d}k_\alpha \left\{K_{\beta\alpha}(Q_\beta - k_\alpha) + K_{\beta\alpha}(-Q_\beta - k_\alpha)\right\}\right]\delta_{\beta\gamma} \tag{3.12}$$

$$= 2t_\alpha\epsilon'_\beta(Q_\beta)\mathcal{Z}_{\alpha\beta}\delta_{\beta\gamma} \tag{3.13}$$

where $\mathcal{Z}_{\alpha\beta} \equiv Z_{\alpha\beta}(Q_\beta)$ is the dressed charge matrix defined in (3.2). Therefore, we obtain the finite size correction $\Delta\varepsilon \equiv \tilde{\varepsilon} - \varepsilon$ due to the small shift of δ_α as

$$\Delta\varepsilon = \frac{1}{2}\sum_{\alpha\beta\gamma}\delta_\beta \left.\frac{\partial^2\tilde{\varepsilon}_\alpha}{\partial\delta_\beta\partial\delta_\gamma}\right|_{\delta=0}\delta_\gamma$$

$$= \sum_{\alpha\beta} t_\alpha(\mathcal{Z}V\mathcal{Z}^{\mathrm{T}})_{\alpha\beta}t_\beta\delta_\beta\delta_\gamma \tag{3.14}$$

where $V_{\alpha\beta} = v_\alpha\delta_{\alpha\beta}$, and v_α is the Fermi velocity defined by $v_\alpha = \epsilon'_\alpha(Q_\alpha)/\rho_\alpha(Q_\alpha)$. To derive the second line, we have used equations (3.4) and (3.13).

The remaining task is to determine the relation between δ_α and A. The result is quite simple (see equation (3.23)), but still non-trivial, which depends on types of external fields. Here we briefly depict how to obtain this relation in a rather general way [15]. For this purpose, let us first introduce

$$i_\alpha(k_\alpha) = \int^{k_\alpha} \mathrm{d}k\ \rho_\alpha(k) \qquad \tilde{i}_\alpha(k_\alpha) = \int^{k_\alpha} \mathrm{d}k\ \tilde{\rho}_\alpha(k) \tag{3.15}$$

$$G_{\alpha\beta}(k_\alpha - k'_\beta) = \int^{k_\alpha} \mathrm{d}k\ g'_{\alpha\beta}(k - k'_\beta). \tag{3.16}$$

When $A = 0$, equation (2.7) is integrated as

$$i_\alpha(k_\alpha) = t_\alpha k_\alpha - \sum_\beta \int_{-Q_\beta}^{Q_\beta} \mathrm{d}k'_\beta\ G_{\alpha\beta}(k_\alpha - k'_\beta)\rho_\beta(k'_\beta). \tag{3.17}$$

Note that the first term on the right-hand side is regarded as a bare momentum of the system, which can be written as $p^0_\alpha \equiv t_\alpha k_\alpha$. In the presence of the vector potential, the momentum is shifted as $p^0_\alpha \to p^0_\alpha - At_\alpha$. Then equation (3.17) is modified to

$$\tilde{i}_\alpha(k_\alpha) + d_\alpha = t_\alpha k_\alpha - \sum_\beta \int_{-Q_\beta+\delta_\beta}^{Q_\beta+\delta_\beta} \mathrm{d}k'_\beta\ G_{\alpha\beta}(k_\alpha - k'_\beta)\tilde{\rho}_\beta(k'_\beta) \tag{3.18}$$

where we denote $d_\alpha \equiv At_\alpha$. Note that the following calculation can also be applied for other types of external fields if we suitably take d_α different from At_α. Now introduce $\tilde{k}_\alpha = \tilde{k}_\alpha(k_\alpha)$ such that $i_\alpha(k_\alpha) = \tilde{i}_\alpha(\tilde{k}_\alpha)$, then we find $\tilde{k}_\alpha(Q_\alpha) = Q_\alpha + \delta_\alpha$ and

$$\rho_\alpha(k_\alpha)\mathrm{d}k_\alpha = \tilde{\rho}_\alpha(\tilde{k}_\alpha)\mathrm{d}\tilde{k}_\alpha. \tag{3.19}$$

By the use of these relations, we can change the integral variables into $\tilde{k}_\alpha$ in (3.18). By subtracting both sides of (3.17) from (3.18) and expanding $G_{\alpha\beta}(\tilde{k}_\alpha - \tilde{k}'_\beta)$ up to first order in $(\tilde{k} - k)$, we get

$$F_\alpha(k_\alpha) = d_\alpha - \sum_\beta \int_{-Q_\beta}^{Q_\beta} \mathrm{d}k'_\beta\ g'_{\alpha\beta}(k_\alpha - k'_\beta)F_\beta(k'_\beta) \tag{3.20}$$

where F_α is defined by [15]

$$F_\alpha(k_\alpha) = (\tilde{k}_\alpha - k_\alpha)\rho_\alpha(k_\alpha). \tag{3.21}$$

The formal solution to (3.20) is given by

$$F_\alpha(k_\alpha) = \sum_\beta d_\beta Z_{\beta\alpha}(k_\alpha). \tag{3.22}$$

As is stressed above, this formula is valid for arbitrary types of d_α. By explicitly substituting $d_\alpha \equiv At_\alpha$, and comparing (3.21) and (3.22) by the use of (3.4), we get the simple relation

$$\tilde{k}_\alpha - k_\alpha = A = \delta_\alpha \tag{3.23}$$

namely that momentum shifts occur not only for the charge sector but for all indices α.

Consequently, by combining (3.14) and (3.23), we end up with the final formula for the energy increment as

$$\Delta\varepsilon = \sum_{\alpha\beta} t_\alpha (ZVZ^{\mathrm{T}})_{\alpha\beta} t_\beta A^2. \tag{3.24}$$

This completes the calculation of finite-size corrections due to the static vector potential.

4. Transport coefficients, effective charge and mass

4.1. Conductivity

According to equation (3.24) for the response to the external vector potential, we can obtain the charge stiffness as

$$D_{\mathrm{c}} = \sum_{\alpha\beta} t_\alpha (ZVZ^{\mathrm{T}})_{\alpha\beta} t_\beta. \tag{4.1}$$

An important point is that D_{c} is directly related to the conductivity through the relation

$$\mathrm{Re}\,\sigma(\omega) = e^2 D_{\mathrm{c}} \delta(\omega) \qquad \text{at } \omega = 0 \tag{4.2}$$

according to the linear response theory [11, 12].

In much of the literature up to now, correlation effects on D_{c} have been considered to modify only the effective transport mass m^* [11–13]. However, the transport mass is not sufficient to describe the correlation effects on transport properties, because quasiparticles such as holons in one dimension can carry not only the effective mass but also the effective charge. Therefore, in place of the ordinary interpretation [12, 13], we propose the following natural expression for the conductivity in terms of the effective charge e^* and effective mass m^*:

$$\mathrm{Re}\,\sigma(w) = \frac{\pi e e^* n_{\mathrm{c}}}{m^*} \delta(\omega) \qquad \text{at } \omega = 0 \tag{4.3}$$

where n_{c} is the density of charged particles. The charge stiffness calculated in (4.1) is then related to D_{c} such that

$$D_{\mathrm{c}} \propto \frac{e^*}{m^*}. \tag{4.4}$$

So, the charge stiffness (or conductivity) is not sufficient to derive the effective charge and mass, and another quantity is necessary to determine them. We will show that the effective charge can be derived from the conductance.

4.2. Conductance

It is known that the fractional charge enters in the conductance for the finite system in one dimension. Although it is not easy to calculate the conductance without wavefunctions, we can compute it by taking into account a universal property of Luttinger liquids, i.e. *the conductance is determined solely by the correlation exponent for the charge sector.* To this end, we first define the critical exponent for the charge density correlation function in the asymptotic region:

$$\langle \rho(x)\rho(0)\rangle \sim \exp(2ik_{\mathrm{F}}^{(\mathrm{c})}x)x^{-2Mf_{\mathrm{c}}} \tag{4.5}$$

where $k_{\mathrm{F}}^{(\mathrm{c})} = \pi n_{\mathrm{c}}$ is the 'Fermi momentum' for the charge sector, and is usually given by $k_{\mathrm{F}}^{(\mathrm{c})} = Mk_{\mathrm{F}}$ in terms of the ordinary Fermi momentum k_{F}. The correlation exponent f_{c} is a function of statistical interactions, which is normalized to reproduce $f_{\mathrm{c}} = 1$ for non-interacting systems. An important point is that the conductance G_{c} can be determined by f_{c} universally [16]:

$$G_{\mathrm{c}} = M\frac{e^2}{h}f_{\mathrm{c}}. \tag{4.6}$$

This formula can be deduced by observing that the conductance is controlled only by the charge degrees of freedom†. Renormalizing e by f_{c} such that $G_{\mathrm{c}} = Mee^*/h$, we can naturally define the effective charge as

$$e^* = f_{\mathrm{c}}e. \tag{4.7}$$

Therefore the remaining task is to obtain the critical exponent f_{c} in terms of statistical interactions. Following a path similar to that in the the last section, we can derive the exponent f_{c} through conformal-field-theory analysis of the finite-size spectrum which has already been computed in [6]. We thus obtain the renormalizarion factor for the fractional charge

$$f_{\mathrm{c}} = \frac{1}{M}\sum_{\alpha\beta} t_\alpha(\mathcal{Z}\mathcal{Z}^{\mathrm{T}})_{\alpha\beta}t_\beta \tag{4.8}$$

in terms of statistical interactions which are implicitly incorporated in the dressed charge matrix. By applying the above formulae to (4.4), we can also extract the effective transport mass m^*, which turns out to be inversely proportional to the velocity v. The expressions (4.1), (4.4), (4.7), (4.8) are the main results of this paper.

One can see that the effective charge is determined *solely by the statistical interactions*, whereas the effective mass also depends on non-universal quantities such as the velocity. We wish to emphasize that the formula for the fractional charge (4.7) with (4.8) is universal, which holds generally for multicomponent Luttinger liquids.

5. Applications

5.1. Ideal fractional exclusion statistics

One of the most remarkable applications of exclusion statistics is that for the ideal case, in which the statistical interaction is given in a simple form [2–6],

$$g_{\alpha\beta}(k_\alpha - k'_\beta) = G_{\alpha\beta}\delta(k_\alpha - k'_\beta). \tag{5.1}$$

† The exponent g without randomness in [16] corresponds to f_{c} in this paper.

This model is known to have a close relationship to interesting quantum systems such as the FQHE, $1/r^2$ systems, etc.

If we take a bare dispersion as $\epsilon_\alpha^0(k) = \epsilon^0(k)t_\alpha$ with $\epsilon(k) = k^2/2$, the ground-state configuration is $Q_1 = Q_2 = \cdots = Q_M \equiv Q$, and the distribution functions are obtained as

$$\rho_\alpha = \sum_\beta G^{-1}_{\alpha\beta} t_\beta. \tag{5.2}$$

The number of charged particles is

$$n_c = 2Q\nu \qquad \nu = \sum_{\alpha\beta} t_\alpha G^{-1}_{\alpha\beta} t_\beta \tag{5.3}$$

where the quantity ν is related to the compressibility as $\kappa_c = 4\nu^2/n_c$, which is a one-dimensional analogue of the filling factor in the FQHE. In the present model, the Fermi velocities for each excitation take the same values

$$v_1 = v_2 = \cdots = v_M \equiv v = \frac{n_c}{2\nu}. \tag{5.4}$$

As shown in the appendix, one finds a simple relation between the dressed charge and the statistical interaction, namely

$$\mathcal{Z}\mathcal{Z}^{\mathrm{T}} = G^{-1}. \tag{5.5}$$

Consequently, the renormalization factor (4.8) for the fractional charge $e^*/e = f_c$ is expressed as

$$f_c = \frac{1}{M} \sum_{\alpha\beta} t_\alpha (G^{-1})_{\alpha\beta} t_\beta. \tag{5.6}$$

It should be noted that the fractional charge is now explicitly obtained only in terms of the statistical parameters $G_{\alpha\beta}$ for exclusion statistics. Since the charge stiffness is given by $D_c = n_c/2$, the enhancement factor for the transport mass in this case is derived as

$$m^*/m = f_c \tag{5.7}$$

from which one can see that the enhancement of m^* exactly cancels the renormalization of charge e^*, in accordance with the translational symmetry.

Let us now discuss more concrete models for ideal exclusion statistics. For an instructive example, we consider the statistical-interaction matrix G in a hierarchical basis, which is derived from the continuum $1/r^2$ model [17] with $SU(M)$ symmetry [18, 19]. This model is also related to a fundamental series for the hierarchical FQHE [8, 20, 21]. The $M \times M$ matrix for statistical interactions in this case reads

$$G = \begin{pmatrix} 2n+1 & -1 & 0 & & \\ -1 & 2 & -1 & 0 & \\ & & \ddots & & \\ & 0 & -1 & 2 & -1 \\ & & 0 & -1 & 2 \end{pmatrix}. \tag{5.8}$$

From (5.3) we have

$$\nu = \cfrac{1}{2n+1-\cfrac{1}{2-\cfrac{1}{\ddots\,\cfrac{1}{2}}}} = \frac{M}{2Mn+1} \equiv \frac{p}{q} \tag{5.9}$$

which corresponds to the filling factor in the case of the FQHE. Observing this, we see that the matrix G in this model plays a role similar to the flux attachment in Jain's model for the FQHE [22]:

$$G \quad \longleftrightarrow \quad \chi_M \chi_1^{2n} \tag{5.10}$$

where χ_M is the IQH state with the filling factor M, which is attached by $2n$ flux quanta χ_1^{2n}. In fact, one can find the same matrix as (5.8) in the classification scheme in the corresponding Abelian Chern–Simons theory for the FQHE [8, 9, 20, 21]. Therefore it can be seen that the $SU(M)$ $1/r^2$ model bears a close relationship to the hierarchical FQHE with $\nu = M/(2Mn+1)$.

According to equations (5.6) and (5.7), it turns out that the effective charge and mass are given by taking $t^{\mathrm{T}} = (1, 0, \cdots, 0)$

$$e^* = \frac{e}{2Mn+1} = \frac{e}{q} \tag{5.11}$$

$$m^* = \frac{m}{2Mn+1} = \frac{m}{q}. \tag{5.12}$$

Note that the expression for the effective charge (5.11) is actually in accordance with that for the FQHE. In particular, for the one-component case $G = g$, we have $e^*/e = m^*/m = 1/g$. Note that for free systems, i.e. $n = 0$, we have $e^*/e = m^*/m = 1$.

5.2. Correlated electron systems

It is also instructive to apply the results to one-dimensional correlated electron systems. In order to fully describe interacting electron systems in the whole energy range, it is necessary to consider statistical interactions that depend on the momentum in a complicated way [23]. However, if we restrict ourselves to the low-energy conformal limit, we can still use the idea of *ideal exclusion statistics*. In such a low-energy region, the critical behaviour is described by the Luttinger liquid theory, in other words, by $c = 1$ conformal field theory. In this case, we can introduce the 2×2 matrix $G_{\alpha\beta}$ for the ideal statistics in (5.1) in terms of Luttinger liquid parameters. Since this model has two kinds of elementary excitations, i.e. spinon and holon, which have two different velocities, v_{s} and v_{c}, we can choose the hierarchical basis as a natural one. By analysing exactly solved models or Tomonaga–Luttinger models, the matrix for statistical interaction can be deduced as [6]

$$G = \begin{pmatrix} \dfrac{K_\rho + 1}{2K_\rho} & -1 \\ -1 & 2 \end{pmatrix} \tag{5.13}$$

where $K_\rho = Z_{11}^2/2$ is the critical exponent for the $4k_{\mathrm{F}}$ oscillation piece in the density correlation function. Note that G_{11} is related to the charge degrees of freedom, G_{22} to the spin degrees of freedom, and the off-diagonal elements are regarded as mutual statistics. It is to be emphasized here that equation (5.13) is the universal formula for correlated electron systems.

According to (4.7), the effective charge is given by substituting $t^{\mathrm{T}} = (1, 0)$

$$e^*/e = K_\rho \tag{5.14}$$

which reproduces the known results for the conductance in Luttinger liquids [16]. Also, from (4.1), the charge stiffness is found to be

$$D_{\mathrm{c}} = 2v_{\mathrm{c}}K_\rho \tag{5.15}$$

from which we obtain the effective mass using (4.4) as

$$m^*/m = v_F/v_c. \tag{5.16}$$

where v_F is the Fermi velocity for non-interacting electrons.

For exactly solvable electron models such as the Hubbard model and the supersymmetric t–J model, the critical exponent K_ρ and the velocity of holons were calculated exactly as functions of the interaction and the electron density [24–26]. So, we can discuss the effective charge and mass for these models. As for the Hubbard model, K_ρ decreases from 1 to $\frac{1}{2}$ as the Coulomb interaction is increased [26], hence the effective charge decreases with the increase of the interaction, as stressed by Ogata and Fukuyama [16]. Near half filling, the effective charge always takes $e/2$ as far as the Coulomb interaction exists. In the case of the supersymmetric t–J model [24, 25], the effective charge is $e/2$ near half filling, but as the electron density decreases, it continuously increases and reaches the non-interacting value $e^* = e$ in the low density limit. Also, we can discuss the effective mass for electron systems. The results are essentially the same as those discussed previously [13]: the effective mass has a divergence property near half filling both for the Hubbard model and the supersymmetric t–J model, reflecting the metal–insulator transition.

6. Summary and discussions

In summary, we have obtained the transport coefficients, effective charge and effective mass for multicomponent quantum systems obeying fractional exclusion statistics. Their explicit relation to the statistical interaction has been derived in equations (4.1)–(4.8) for general systems obeying (2.1). We have applied the results for the cases with ideal statistics as well as for the conformal limit of electron systems. It has also been pointed out that the statistical interaction derived from $SU(N)$ $1/r^2$ models is closely related to Jain's construction (or the corresponding Chern–Simons theory) for the hierarchical FQHE.

It is instructive to note that the effective charge in the *ideal* case is expressed in an extremely simple form (5.6) or (5.11) in terms of the statistical interaction $G_{\alpha\beta}$, implying that the fractional charge in the ideal case directly reflects the fractional statistics. In fact, we find an alternative way of deriving (5.11) using only the definition of fractional exclusion statistics. We briefly summarize how to get them intuitively. In the ideal case, the definition (2.1) reads

$$\frac{\partial N_\alpha}{\partial D_\beta} = -G^{-1}_{\alpha\beta} \qquad \text{at } k_\alpha = k_\beta. \tag{6.1}$$

Now imagine the ground-state configuration and make a hole to excite the system. The above equation implies that if we make a β-hole, the number of α-particles decreases by an amount $G^{-1}_{\alpha\beta}$. Then, how many charged particles decrease in all? The answer is $\sum_\alpha t_\alpha G^{-1}_{\alpha\beta}$. Now make a hole at a sector α, i.e. $t^T = (0, \ldots, 1, 0, \ldots, 0)$ in the hierarchical basis. This corresponds to $t^T = (0, \ldots, 1, 1, \ldots, 1)$ in the symmetric basis, which may create $M + 1 - \alpha$ holes of electrons. Therefore, making a hole with unit charge corresponds to removing particles with the charge

$$e^*_\alpha = \frac{e}{M + 1 - \alpha} \sum_\beta G^{-1}_{\beta\alpha} \tag{6.2}$$

which in turn defines the effective charge of the excitation. If we adopt G in (5.8), then we have

$$e^*_\alpha/e = 1/q \tag{6.3}$$

where q is defined in (5.9). This result coincides with (5.11).

Finally, another remark is in order. In section 3, we have defined the effective charge (4.7) apart from the trivial degeneracy M. However, it may be possible to include such a factor in the definition of the charge. In this definition, equations (5.6), (5.11), (5.12) are modified by the factor M, and equation (6.2) should be replaced by

$$\tilde{e}^*_\alpha = e \sum_\beta G^{-1}_{\beta\alpha} \tag{6.4}$$

and therefore, equation (6.3) is modified as

$$\tilde{e}^*_\alpha / e = (M + 1 - \alpha)/q \tag{6.5}$$

i.e. explicitly, $p/q, (p-1)/q, \ldots, 1/q$. This definition for the fractional charge corresponds to that used in [8, 9].

Acknowledgments

This work is supported by a Grant-in-Aid from the Ministry of Education, Science and Culture.

Appendix

In the case of ideal exclusion statistics, there exists a simpler way of deriving equation (5.5) without calculating the dressed charge (3.2). We briefly depict this convenient method below. Consider the present system without external fields. Then there are two kinds of elementary excitation, i.e. excitation which changes the number of particles, and that which carries the large momentum. Following the techniques of the dressed charge matrix [14], the excitation spectrum was explicitly evaluated in [6] as

$$\Delta\varepsilon = (v/4)\boldsymbol{n}^{\mathrm{T}}(\mathcal{Z}\mathcal{Z}^{\mathrm{T}})^{-1}\boldsymbol{n} + v\boldsymbol{d}^{\mathrm{T}}(\mathcal{Z}\mathcal{Z}^{\mathrm{T}})\boldsymbol{d}$$

where the vectors $\boldsymbol{n}$ and $\boldsymbol{d}$ denote the quantum numbers for excitations, labelling the change of particle number and the momentum transfer, respectively. Note that the excitation which we seek in (3.24) corresponds to the excitation specified by $\boldsymbol{d}$. From the above formula, one can see that these two kinds of excitations are related to each other reflecting modular invariance. So, we can easily deduce the excitations for the $\boldsymbol{d}$-sector once we can calculate those for the $\boldsymbol{n}$-sector. The calculation for the latter excitation is much simpler than the former. Let us then calculate the latter by changing $Q_\alpha \to Q_\alpha + \Delta Q_\alpha$. Then both n_α and $\Delta\varepsilon$ are given by functions of ΔQ_α, and a simple calculation gives $\Delta\varepsilon$ as a function of n_α such that $\Delta\varepsilon = (v/4)\boldsymbol{n}^{\mathrm{T}}G\boldsymbol{n}$. Comparing these results, we end up with equation (5.6).

References

[1] Prange R E and Girvin S M (ed) 1990 *The Quantum Hall Effect* (Berlin: Springer)
Stone M (ed) 1992 *Quantum Hall Effect* (Singapore: World Scientific)
[2] Haldane F D M 1991 *Phys. Rev. Lett.* **67** 937
[3] Murthy M V N and Shankar R 1994 *Phys. Rev. Lett.* **72** 3629; 1994 *Preprint* IMSc-94/24, cond-mat/9404096
[4] Wu Y-S 1994 *Phys. Rev. Lett.* **73** 922
Bernard D and Wu Y-S 1994 *Preprint* SPhT-94-043, UU-HEP/94-03, cond-mat/9404025
[5] Nayak C and Wilczek F 1994 *Phys. Rev. Lett.* **73** 2740
[6] Fukui T and Kawakami N 1995 *Phys. Rev.* B **51** 5239
[7] Bhaduri R K, Murthy M V N and Shankar R 1994 *Preprint* IMSc-94/42, cond-mat/9411092
[8] Wen X G and Zee A 1992 *Phys. Rev.* B **46** 2299

[9] Wen X G 1992 *Int. J. Mod. Phys.* **6** 1711
[10] Yang C N 1967 *Phys. Rev. Lett.* **19** 1312
Sutherland B 1968 *Phys. Rev. Lett.* **20** 98
[11] Kohn W 1964 *Phys. Rev.* **133** 171
[12] Shastry B S and Sutherland B 1990 *Phys. Rev. Lett.* **65** 243
[13] Kawakami N and Yang S-K 1991 *Phys. Rev.* B **44** 7844
[14] Izergin A G, Korepin V E and Reshetikhin N Y 1989 *J. Phys. A: Math. Gen.* **22** 2615
[15] Korepin V E, Bogoliubov N M and Izergin A G 1993 *Quantum Inverse Scattering Method and Correlation Functions* (Cambridge: Cambridge University Press)
[16] Apel W and Rice T M 1982 *Phys. Rev.* B **26** 7063
Kane G L and Fisher M P A 1992 *Phys. Rev. Lett.* **68** 1220
Ogata M and Fukuyama H 1994 *Phys. Rev. Lett.* **73** 468
[17] Calogero F 1969 *J. Math. Phys.* **10** 2197
Sutherland B 1971 *J. Math. Phys.* **12** 246, 256; 1971 *Phys. Rev.* A **4** 2019; 1971 *Phys. Rev.* A **5** 1372
[18] Haldane F D M 1988 *Phys. Rev. Lett.* **60** 635
Shastry B S 1988 *Phys. Rev. Lett.* **60** 639
[19] Kuramoto Y and Yokoyama H 1991 *Phys. Rev. Lett.* **67** 1338
Kawakami N 1992 *Phys. Rev.* B **46** 1005
Ha Z N C and Haldane F D M 1992 *Phys. Rev.* B **46** 9359
Hikami K and Wadati M 1993 *J. Phys. Soc. Japan* **62** 469
Minahan J A and Polychronakos A P 1993 *Phys. Lett.* **302B** 265
[20] Blok B and Wen X G 1990 *Phys. Rev.* B **42** 8133
[21] Read N 1990 *Phys. Rev. Lett.* **65** 1502
[22] Jain J K 1990 *Phys. Rev.* B **41** 7653
[23] Hatsugai Y, Kohmoto M, Koma T and Wu Y-S 1995 *Preprint* cond-mat/9504112
[24] Schlottmann P 1987 *Phys. Rev.* B **36** 5177
[25] Kawakami N and Yang S-K 1991 *J. Phys.: Condens. Matter* **3** 5983
[26] Schulz H J 1990 *Phys. Rev. Lett.* **64** 2831
Kawakami N and Yang S-K 1990 *Phys. Lett.* **148A** 359
Frahm H and Korepin V E 1990 *Phys. Rev.* B **42** 10553